

Rapidly convergent zeta series for $\zeta(2n+1)$ and $\beta(2n)$ and their generalization

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Abstract

In a very recent work on Euler-type formulae for even Dirichlet beta values, i.e. $\beta(2n)$, n being a positive integer, I have derived an exact closed-form expression for a family of zeta series. From this family, the analytical results I have found for certain related series led me to conjecture exact expressions for two other families of zeta series. Here in this work, I make use of a classical Wilton's formula to prove the simpler conjecture. The other conjecture is proved in two independent forms. The comparison of this second result with a well-known theorem by Srivastava yields a new identity relating $\beta(2n)$ to the first derivatives of the Riemann zeta and the Hurwitz zeta functions. Indeed, the generalization of the mentioned results for zeta series has led me to detect an error in a theorem by Katsurada. The corrected version of this theorem is presented here.

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1. Introduction

For real values of s , $s > 1$, the Riemann zeta function is defined as $\zeta(s) := \sum_{m=1}^{\infty} 1/m^s$ (note that this function has a singularity at $s = 1$, which corresponds to the divergence of the harmonic series). For integer values of s , $s > 1$, one has a well-known formula first proved by Euler (see Ref. [2] and references therein), namely

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} B_{2n}}{(2n)!} \pi^{2n}, \quad (1)$$

n being a positive integer. Here, B_n are Bernoulli numbers, i.e. the rational coefficients of $z^n/n!$ in the Taylor expansion of $z/(e^z - 1)$, $|z| < 2\pi$. For $\zeta(2n+1)$, on the other hand, no analogous expression is currently known.

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This scenario has a ‘reverse’ counterpart on the values of the Dirichlet beta function, defined as $\beta(s) := \sum_{k=0}^{\infty} (-1)^k / (2k+1)^s$, $s > 0$. For positive integer values of s , the following analogue of Eq. (1) is well-known:¹

$$\beta(2n+1) = (-1)^n \frac{E_{2n}}{2^{2n+2} (2n)!} \pi^{2n+1},$$

where E_n are the Euler numbers.² For $\beta(2n)$, no finite closed-form expression in terms of elementary functions is known.³ Some progress in this direction was reached by Kölbig (1996), who proved the following identity [4]:

$$\beta(2n) = \frac{\psi^{(2n-1)}(\frac{1}{4})}{2(2n-1)! 4^{2n-1}} - \frac{(2^{2n}-1) |B_{2n}|}{2(2n)!} \pi^{2n}, \quad (2)$$

where $\psi^{(n)}(x)$ is the polygamma function, i.e. the n -th derivative of $\psi(x)$, the digamma function.⁴ Although this relation resembles the Euler’s formula, it is currently not known how to express the numbers $\psi^{(2n-1)}(\frac{1}{4})$ in terms of other known mathematical constants. In a recent work, I have succeeded in applying the Dancs and He series expansion method (see Ref. [1]) to find similar formulae for $\beta(2n)$ [5]. I could then prove that [5]

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k(2k+1) \dots (2k+2n-1)} \left(\frac{1}{2^{2k}} - \frac{1}{4^{2k}} \right) = (-1)^n \frac{2^{2n-2}}{\pi^{2n-1}} \beta(2n) + \frac{n}{(2n)!} \ln 2 + \frac{1}{2} \sum_{m=1}^{n-1} (-1)^m \frac{2^{2m}-1}{\pi^{2m}} \frac{\zeta(2m+1)}{(2n-2m-1)!}, \quad (3)$$

where n is a positive integer and $H_n := \sum_{k=1}^n \frac{1}{k}$ is the n -th harmonic number. Since both series

$$\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{2^{2k}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{4^{2k}}$$

converge absolutely, the zeta series in Eq. (3) can be written as the difference of these two individual series. However, the best I could do there in Ref. [5] was to investigate the pattern of the analytical results found for the first zeta series above, for small values of n . This procedure led me to *conjecture* a formula for this series which should be valid for any positive integer n . Of course, as follows from Eq. (3), an eventual proof of this formula would furnish an analytical expression to the other individual zeta series.

¹Note that this formula remains valid for $n = 0$, since $E_0 = 1$ and $\beta(1) = \pi/4$.

²Namely, the (integer) coefficients of $z^n/n!$ in the Taylor expansion of $\text{sech}(z)$, $|z| < \pi/2$.

³Note that this includes the famous Catalan’s constant $G := \beta(2)$.

⁴The function $\psi(x)$, in turn, is defined as the logarithm derivative of $\Gamma(x)$.

Here in this work, I make use of a classical Wilton's formula to prove the simpler conjecture. The second conjecture is proved in two independent forms: by using Eq. (3) and by taking into account a formula by Srivastava and Tsumura (2000). The comparison of this result with a well-known theorem by Srivastava (1998) yields a new identity relating $\beta(2n)$ to the first derivatives of the Riemann zeta and the Hurwitz zeta functions. Finally, on aiming at a generalization of these formulae I investigate the substitution of the fractions $(1/2)^{2k}$ and $(1/4)^{2k}$ by x^{2k} , $x \in \mathbb{R}$, $|x| \leq 1$, in those zeta series, which has allowed me to detect and correct an error in a theorem by Katsurada (1999).

2. Zeta series for $\zeta(2n+1)$ and $\beta(2n)$

The proof for the first conjecture raised in Ref. [5], involving the first individual zeta series above, is based upon the following Wilton's formula, which yields a rapidly convergent series representation for odd zeta values [10].

Lemma 1 (Wilton's formula). *Let n be a positive integer. Then*

$$\begin{aligned} \frac{\zeta(2n+1)}{(-1)^{n-1} \pi^{2n}} &= \frac{H_{2n+1} - \ln \pi}{(2n+1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ &\quad + \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k-1}}. \end{aligned}$$

PROOF. For a skeleton of the proof, see the original work of Wilton [10]. For a more complete proof, see Sec. 4.2 (in particular, pp. 412–413) of Ref. [9], a systematic collection of zeta series published by Srivastava and Choi. \square

This lemma allows us to prove the first conjecture of Ref. [5] (see its Eq. (29)).

Theorem 1 (First zeta series). *Let n be a positive integer. Then*

$$\sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{(2k+2n-1)!} \left(\frac{1}{2}\right)^{2k} = \frac{1}{2} \left[\frac{\ln \pi - H_{2n-1}}{(2n-1)!} + \sum_{m=1}^{n-1} (-1)^{m+1} \frac{\zeta(2m+1)}{\pi^{2m} (2n-2m-1)!} \right].$$

PROOF. From Lemma 1, we know that, for any positive integer n ,

$$\begin{aligned} \frac{(-1)^n \zeta(2n+1)}{\pi^{2n}} &= \frac{\ln \pi - H_{2n+1}}{(2n+1)!} - \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ &\quad - 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}}. \end{aligned}$$

By isolating the last term (i.e., the zeta series), one has

$$2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} = \frac{\ln \pi - H_{2n+1}}{(2n+1)!} - \sum_{k=1}^n \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}}.$$

By substituting $n = \ell - 1$ in the above equation, one finds

$$2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2\ell-1)!} \frac{\zeta(2k)}{2^{2k}} = \frac{\ln \pi - H_{2\ell-1}}{(2\ell-1)!} - \sum_{k=1}^{\ell-1} \frac{(-1)^k}{(2\ell-2k-1)!} \frac{\zeta(2k+1)}{\pi^{2k}}.$$

A division by 2 completes the proof. \square

Now we can use Eq. (3) to prove the conjecture in Eq. (30) of Ref. [5].

Theorem 2 (Second zeta series). *Let n be a positive integer. Then*

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{\left(\frac{1}{4}\right)^{2k}} &= \frac{1}{2} \left[\frac{\ln(\pi/2) - H_{2n-1}}{(2n-1)!} - (-1)^n \left(\frac{2}{\pi}\right)^{2n-1} \beta(2n) \right. \\ &\quad \left. - \sum_{m=1}^{n-1} (-1)^m \left(\frac{2}{\pi}\right)^{2m} \frac{\zeta(2m+1)}{(2n-2m-1)!} \right]. \end{aligned}$$

PROOF (FIRST). From Eq. (3), we know that

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{\left(\frac{1}{2^{2k}}\right)} - \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{\left(\frac{1}{4^{2k}}\right)} \\ &= (-1)^n \frac{2^{2n-2}}{\pi^{2n-1}} \beta(2n) + \frac{n}{(2n)!} \ln 2 + \frac{1}{2} \sum_{m=1}^{n-1} (-1)^m \frac{2^{2m}-1}{\pi^{2m}} \frac{\zeta(2m+1)}{(2n-2m-1)!}. \end{aligned}$$

By substituting the first zeta series at the left-hand side by the analytical expression established in Theorem 1, above, one has

$$\begin{aligned} &\frac{\ln \pi - H_{2n-1}}{(2n-1)!} - \sum_{m=1}^{n-1} (-1)^m \frac{\zeta(2m+1)}{\pi^{2m} (2n-2m-1)!} - 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{\left(\frac{1}{4^{2k}}\right)} \\ &= (-1)^n \left(\frac{2}{\pi}\right)^{2n-1} \beta(2n) + \frac{\ln 2}{(2n-1)!} + \sum_{m=1}^{n-1} (-1)^m \frac{2^{2m}-1}{\pi^{2m}} \frac{\zeta(2m+1)}{(2n-2m-1)!}. \end{aligned}$$

By isolating the remaining zeta series, one finds

$$\begin{aligned} 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{\left(\frac{1}{4^{2k}}\right)} &= \frac{\ln(\pi/2) - H_{2n-1}}{(2n-1)!} - (-1)^n \left(\frac{2}{\pi}\right)^{2n-1} \beta(2n) \\ &\quad - \sum_{m=1}^{n-1} (-1)^m \left(\frac{2}{\pi}\right)^{2m} \frac{\zeta(2m+1)}{(2n-2m-1)!}. \end{aligned}$$

A division by 2 completes the proof. \square

After an extensive search for similar zeta series in literature, I have noted that a zeta series for $\zeta(2n+1)$ derived by Srivastava and Tsumura (2000) could yield an independent proof of Theorem 2.

PROOF (SECOND). Let us take into account the following zeta series for $\zeta(2n+1)$, n being a positive integer, introduced by Srivastava and Tsumura [8] [see also Ref. [9], p. 421, Eq. (30)]:

$$\begin{aligned} \frac{\zeta(2n+1)}{(-1)^{n-1} (\pi/2)^{2n}} &= \frac{H_{2n+1} - \ln(\pi/2)}{(2n+1)!} + \frac{2^{2n} (2^{2n+2} - 1) \pi}{(2n+2)!} B_{2n+2} \\ &+ \frac{(-1)^{n-1}}{2 (2\pi)^{2n+1}} \zeta\left(2n+2, \frac{1}{4}\right) + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{(\pi/2)^{2k}} \\ &+ 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{4^{2k}}, \end{aligned} \quad (4)$$

where $\zeta(s, a) := \sum_{k=0}^{\infty} 1/(k+a)^s$ is the Hurwitz zeta function. By isolating the only infinite zeta series, one finds, after some algebra,

$$\begin{aligned} 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{4^{2k}} &= \frac{\ln(\pi/2) - H_{2n+1}}{(2n+1)!} - \sum_{k=1}^n \frac{(-1)^k \zeta(2k+1)}{(2n-2k+1)!} \left(\frac{2}{\pi}\right)^{2k} \\ &+ \frac{(-1)^n}{2 (2\pi)^{2n+1}} \zeta\left(2n+2, \frac{1}{4}\right) - \frac{(2^{4n+2} - 2^{2n}) \pi}{(2n+2)!} B_{2n+2}. \end{aligned}$$

By substituting $n = m-1$, one has

$$\begin{aligned} 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2m-1)!} \frac{\zeta(2k)}{4^{2k}} &= \frac{\ln(\pi/2) - H_{2m-1}}{(2m-1)!} - \sum_{k=1}^{m-1} \frac{(-1)^k \zeta(2k+1)}{(2m-2k-1)!} \left(\frac{2}{\pi}\right)^{2k} \\ &+ \frac{(-1)^{m-1}}{2 (2\pi)^{2m-1}} \zeta\left(2m, \frac{1}{4}\right) - \frac{(2^{4m-2} - 2^{2m-2}) \pi}{(2m)!} B_{2m}. \end{aligned} \quad (5)$$

Since $\zeta(n+1, a) = (-1)^{n+1} \psi^{(n)}(a)/n!$ (see, e.g., Eq. (25.11.12) of Ref. [6]), one knows that

$$\zeta\left(2m, \frac{1}{4}\right) = \frac{\psi^{(2m-1)}\left(\frac{1}{4}\right)}{(2m-1)!}. \quad (6)$$

Equation (5) can then be rewritten as

$$\begin{aligned} 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2m-1)!} \frac{\zeta(2k)}{4^{2k}} &= \frac{\ln(\pi/2) - H_{2m-1}}{(2m-1)!} - \sum_{k=1}^{m-1} \frac{(-1)^k \zeta(2k+1)}{(2m-2k-1)!} \left(\frac{2}{\pi}\right)^{2k} \\ &+ \frac{(-1)^{m-1}}{(2\pi)^{2m-1}} \frac{\psi^{(2m-1)}\left(\frac{1}{4}\right)}{2 (2m-1)!} - \frac{(2^{4m-2} - 2^{2m-2}) \pi}{(2m)!} B_{2m}. \end{aligned} \quad (7)$$

From Eq. (2), we know that

$$\frac{\psi^{(2m-1)}\left(\frac{1}{4}\right)}{2 (2m-1)!} = 2^{4m-2} \beta(2m) + (-1)^{m-1} 2^{4m-3} \frac{(2^{2m} - 1) B_{2m}}{(2m)!} \pi^{2m}.$$

Note that we have changed $|B_{2m}|$ by $(-1)^{m-1} B_{2m}$ in the last term. By substituting this on the right-hand side of Eq. (7), one finds

$$2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2m-1)!} \frac{\zeta(2k)}{4^{2k}} = \frac{\ln(\pi/2) - H_{2m-1}}{(2m-1)!} - \sum_{k=1}^{m-1} \frac{(-1)^k \zeta(2k+1)}{(2m-2k-1)!} \left(\frac{2}{\pi}\right)^{2k} \\ + \frac{(-1)^{m-1} 2^{2m-1} \beta(2m) + (2^{4m-2} - 2^{2m-2}) \pi^{2m} B_{2m}/(2m)!}{\pi^{2m-1}} - \frac{(2^{4m-2} - 2^{2m-2}) \pi}{(2m)!} B_{2m}.$$

By canceling the terms containing B_{2m} , this equation simplifies to

$$2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2m-1)!} \frac{\zeta(2k)}{4^{2k}} = \frac{\ln(\pi/2) - H_{2m-1}}{(2m-1)!} - \sum_{k=1}^{m-1} \frac{(-1)^k \zeta(2k+1)}{(2m-2k-1)!} \left(\frac{2}{\pi}\right)^{2k} \\ - (-1)^m \left(\frac{2}{\pi}\right)^{2m-1} \beta(2m).$$

A division by 2 completes the proof. \square

It is interesting to compare the formula in Theorem 2 to a lengthier one derived by Srivastava (1998) in Ref. [7]. This leads us to the following identity.

Theorem 3 (Identity for $\beta(2m)$). *Let m be a positive integer and $\zeta'(s, a)$ be the first derivative of the Hurwitz zeta function with respect to s . Then*

$$(-1)^m \left(\frac{2}{\pi}\right)^{2m-1} \beta(2m) = 2 (2^{2m-2} - 1) \frac{B_{2m}}{(2m)!} \ln 2 - \frac{2^{2m-1} - 1}{(2m-1)!} \zeta'(1-2m) \\ - \frac{2^{4m-1}}{(2m-1)!} \zeta'\left(1-2m, \frac{1}{4}\right).$$

PROOF. Let us take into account the following formula derived by Srivastava (see Ref. [7], also Ref. [9], p. 416, Eq. (2)):

$$\frac{\zeta(2n+1)}{(-1)^{n-1} (\pi/2)^{2n}} = \frac{H_{2n+1} - \ln(\pi/2)}{(2n+1)!} + \frac{2(4^n - 1)}{(2n+2)!} B_{2n+2} \ln 2 \\ - \frac{2^{2n+1} - 1}{(2n+1)!} \zeta'(-2n-1) - \frac{2^{4n+3}}{(2n+1)!} \zeta'\left(-2n-1, \frac{1}{4}\right) \\ + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{(\pi/2)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{4^{2k}}.$$

A substitution of n by $m-1$ in this formula, followed by a comparison to the formula established in Theorem 2, completes the proof. \square

Although the identity in Theorem 3 cannot be found in this form in literature, one can use Eq. (2), together with Eq. (6), to write it in terms of $\zeta(2m, \frac{1}{4})$. After some algebra, one finds

$$\begin{aligned} \frac{(-1)^m \pi}{2(2\pi)^{2m}} \zeta\left(2m, \frac{1}{4}\right) &= [(2^{2m-2} - 1) \ln 2 - 2^{2m-3} (2^{2m} - 1) \pi] \frac{B_{2m}}{(2m)!} \\ &\quad - \frac{2^{2m-1} - 1}{2(2m-1)!} \zeta'(1-2m) - \frac{2^{4m-2}}{(2m-1)!} \zeta'\left(1-2m, \frac{1}{4}\right). \end{aligned} \quad (8)$$

This is one of the main results obtained by Srivastava and Tsumura in Ref. [8] (see also Ref. [9], p. 421, Eq. (28)).

Let us now substitute the fractions $(1/2)^{2k}$ and $(1/4)^{2k}$ in the zeta series treated in Theorems 1 and 2, respectively, by x^{2k} , x being a real number with $|x| \leq 1$, in order to generalize our results.

Theorem 4 (Generalization). *Let n be a positive integer and x be a real number with $|x| \leq 1$. Then*

$$\begin{aligned} \zeta(2n+1) - \frac{1}{2\pi x} \sum_{\ell=1}^{\infty} \frac{\sin(2\pi\ell x)}{\ell^{2n+2}} &= (-1)^{n-1} (2\pi x)^{2n} \left[\frac{H_{2n+1} - \ln(2\pi x)}{(2n+1)!} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} (-1)^k \frac{\zeta(2k+1)}{(2n-2k+1)! (2\pi x)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{(2n+2k+1)!} x^{2k} \right]. \end{aligned}$$

PROOF. It is enough to reconsider the proof presented in Ref. [3] by Katsurada for his Theorem 2. That proof, presented there on pp. 84–85, consists in to manipulate certain infinite series according to a Mellin transform technique. This yields two analytical expressions for the same function, namely

$$J(x) = -\frac{H_{2n+1} - \ln(2\pi x)}{2(2n+1)!} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k(2k+1) \dots (2k+2n+1)} x^{2k} \quad (9)$$

and

$$\begin{aligned} J(x) &= \frac{1}{2} (2\pi i x)^{-2n} \zeta(2n+1) + \pi x (2\pi i x)^{-2n-2} \sum_{\ell=1}^{\infty} \frac{\sin(2\pi\ell x)}{\ell^{2n+2}} \\ &\quad + \frac{1}{2} \sum_{k=1}^{n-1} \frac{\zeta(2k+1)}{(2n-2k+1)!} (2\pi i x)^{-2k}, \end{aligned} \quad (10)$$

x being a real with $|x| \leq 1$.

The first expression comes from the integral⁵

$$J(x) = \frac{1}{4i} \int_{(\sigma_0)} \cot\left(\frac{\pi s}{2}\right) \zeta(s) \frac{x^s}{s(s+1) \dots (s+2n+1)} ds,$$

⁵In the derivation of Eq. (9), one uses the facts that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2} \ln(2\pi)$.

where the path (σ_0) consists of the vertical straight-line from $\sigma_0 - i\infty$ to $\sigma_0 + i\infty$, σ_0 being a constant satisfying $-\frac{1}{2} < \sigma_0 < 0$. The other expression is found by applying the Riemann functional equation, namely

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s),$$

in the above integral and changing the variable of integration from s to $1-s$.⁶

In fact, every step in the derivation of Eqs. (9) and (10) in the original paper by Katsurada has been carefully reviewed by the author. The only error, which led him to an *incorrect* statement of his Theorem 2, occurs in the last step of his proof, when he compares the expressions for $J(x)$ in Eqs. (9) and (10). Let us then redo this comparison explicitly. By equalizing the right-hand sides of these equations one finds, after a multiplication by -2 , that

$$\begin{aligned} & \frac{H_{2n+1} - \ln(2\pi x)}{(2n+1)!} + 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k(2k+1) \dots (2k+2n+1)} x^{2k} \\ &= -\frac{\zeta(2n+1)}{(-1)^n (2\pi x)^{2n}} - \frac{\sum_{\ell=1}^{\infty} \sin(2\pi \ell x)/(\ell^{2n+2})}{(-1)^{n+1} (2\pi x)^{2n+1}} - \sum_{k=1}^{n-1} \frac{\zeta(2k+1)}{(2n-2k+1)!} \frac{(-1)^k}{(2\pi x)^{2k}}. \end{aligned}$$

On passing the finite sum to the left-hand side, one has

$$\begin{aligned} & \frac{H_{2n+1} - \ln(2\pi x)}{(2n+1)!} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{(2k+2n+1)!} x^{2k} + \sum_{k=1}^{n-1} \frac{\zeta(2k+1)}{(2n-2k+1)!} \frac{(-1)^k}{(2\pi x)^{2k}} \\ &= \frac{\zeta(2n+1)}{(-1)^{n+1} (2\pi x)^{2n}} - \frac{1}{2\pi x} \frac{\sum_{\ell=1}^{\infty} \sin(2\pi \ell x)/(\ell^{2n+2})}{(-1)^{n+1} (2\pi x)^{2n}}. \end{aligned}$$

A multiplication of both sides by $(-1)^{n-1} (2\pi x)^{2n}$ completes the proof. \square

Let us emphasize that our Theorem 4 corrects a sign error in the sine infinite series (the second term) of Theorem 2 of Ref. [3], an important reference in the study of rapidly convergent zeta series.⁷

Indeed, by noting that $\sum_{\ell=1}^{\infty} \sin(2\pi \ell x)/(\ell^{2n+2}) = \text{Cl}_{2n+2}(2\pi x)$ for all real x , where $\text{Cl}_n(\theta) := \Im\{\text{Li}_n(e^{i\theta})\}$ is the Clausen function of order n , we can write the formula in Theorem 4 as

$$\begin{aligned} & \zeta(2n+1) - \frac{\text{Cl}_{2n+2}(2\pi x)}{2\pi x} = (-1)^{n-1} (2\pi x)^{2n} \left[\frac{H_{2n+1} - \ln(2\pi x)}{(2n+1)!} \right. \\ & \left. + \sum_{k=1}^{n-1} (-1)^k \frac{\zeta(2k+1)}{(2n-2k+1)! (2\pi x)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{(2n+2k+1)!} x^{2k} \right]. \quad (11) \end{aligned}$$

⁶The only lemma on p. 84 of Ref. [3], used in the proof of its Theorem 1, is also applied in the last step of the derivation of Eq. (10).

⁷See also Ref. [9], p. 442, Ex. 18.

Since $\text{Cl}_{2n+2}(\pi) = 0$ and $\text{Cl}_{2n+2}(\pi/2) = \beta(2n+2)$, this form allows for prompt, independent proofs of Theorems 1 and 2, respectively, as the reader can easily check. This shows that these theorems are special cases of Theorem 4, as expected.

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